

INCOMPRESSIBLE SURFACES IN LINK COMPLEMENTS

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ABSTRACT. We generalize a theorem of Finkelstein and Moriah and show that if a link L has a $2n$ -plat projection satisfying certain conditions, then its complement contains some closed essential surfaces. In most cases these surfaces remain essential after any totally nontrivial surgery on L .

A link L in S^3 has a $2n$ -plat projection for some n , as shown in Figure 1, where a box on the i -th row and j -th column consists of 2 vertical strings with a_{ij} left-hand half twist; in other words, it is a rational tangle of slope $1/a_{ij}$. See for example [BZ]. Let n be the number of boxes in the even rows, so there are $n - 1$ boxes in the odd rows. Let m be the number of rows in the diagram. It was shown by Finkelstein and Moriah [FM1, FM2] that if $n \geq 3$, $m \geq 5$, and if $|a_{ij}| \geq 3$ for all i, j , then the link exterior $E(L) = S^3 - \text{Int}N(L)$ contains some essential planar surfaces, which can be tubed on one side to obtain closed incompressible surfaces in $E(K)$. In this note we will prove a stronger version of this theorem, showing that $E(L)$ contains some essential surfaces if $n \geq 3$, the boxes on the two ends of the odd rows have $|a_{ij}| \geq 3$, and $a_{ij} \neq 0$ for the boxes which are not on the ends of the rows. We allow $a_{ij} = 0$ for boxes on the ends of the even rows, and there is no restriction on m , the number of rows in the diagram. The argument here provides a much simpler proof to the above theorem of Finkelstein and Moriah. In [FM2] that theorem was applied to show that if L is a knot then all surgeries on L contain essential surfaces. Corollary 2 below generalizes this to the case when L has multiple components, with a mild restriction that each component of L intersects some “allowable” spheres.

We first give some definitions. Let $\alpha = \alpha(a_1, \dots, a_m)$ be an arc running monotonically from the top to the bottom of the $2n$ -plat, such that α is disjoint from the boxes, and on the i -th row there are a_i boxes on the left of α . See Figure 1 for the arc $\alpha(1, 1, 1, 2, 2)$. The arc α is an *allowable path* if (i) each row has at least one box on each side of α , and (ii) α intersects L at $m + 1$ points, (so α intersects L once when passing from one row to another). Note that the leftmost allowable path is $\alpha(1, \dots, 1)$, which has on its left one box from each row.

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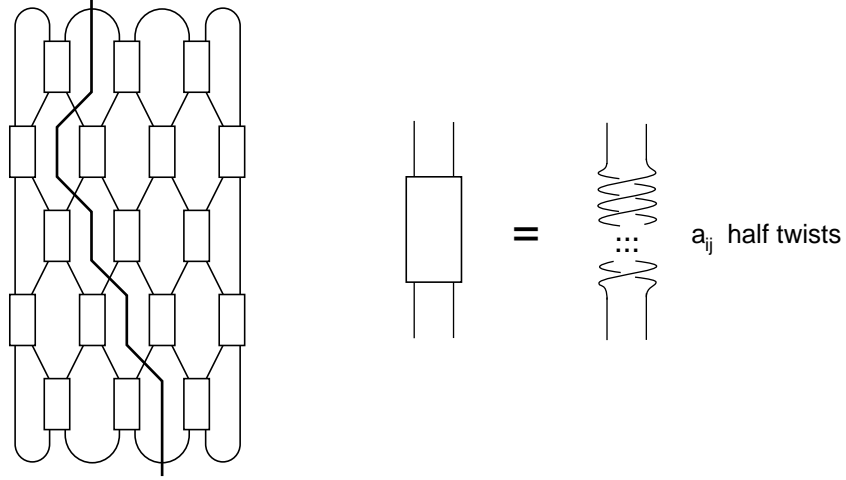


Figure 1

Given an allowable path $\alpha = \alpha(a_1, \dots, a_m)$, we can connect the two ends of α by an arc β disjoint from the projection of L to form a circle, then cap it off by two disks, one on each side of the projection plane, to get a sphere $S = S(a_1, \dots, a_m)$, called an *allowable sphere*. S cuts (S^3, L) into two tangles (B, T) and (B', T') , where (B, T) denotes the one on the left hand side of S . Let $P = P(a_1, \dots, a_m)$ be the planar surface $S \cap E(L)$, which cuts $E(L)$ into two pieces $X = X(a_1, \dots, a_m)$ and $X' = X'(a_1, \dots, a_m)$, with $X = B \cap E(L)$ the one on the left of P . Let $F = F(a_1, \dots, a_m)$ be the surface obtained by tubing P on the left hand side; in other words, F is the component of ∂X containing P , pushed slightly into the interior of $E(L)$. Similarly, denote by $F' = F'(a_1, \dots, a_m)$ the surface obtained by tubing P on the right hand side.

Recall that a properly embedded surface F in a 3-manifold M is an *essential surface* if it is incompressible, ∂ -incompressible, and is not boundary parallel. We define a surface F on the boundary of M to be *essential* if it is incompressible, $M \neq F \times I$, and there is no compressing disk of ∂M which intersects F at a single essential arc in F . Thus if F is properly embedded in M , then it is essential if and only if after cutting along F the two copies of F are essential in the resulting manifold. A 3-manifold M is ∂ -irreducible if ∂M is incompressible in M . Given a set A in M , denote by $N(A)$ a regular neighborhood of A in M .

Theorem 1. *Suppose L has a $2n$ -plat projection such that (i) $n \geq 3$; (ii) $a_{ij} \neq 0$ for $j \neq 0, n$; and (iii) $|a_{ij}| \geq 3$ for i odd and $j = 0$ or $n - 1$. Let $S = S(a_1, \dots, a_m)$ be an allowable sphere. Then $E(L)$ is irreducible, and the surfaces $F = F(a_1, \dots, a_m)$ and $F' = F'(a_1, \dots, a_m)$ are essential in $E(L)$.*

Let $L = L_1 \cup \dots \cup L_k$ be a k component link, let $r = (r_1, \dots, r_k)$ be a set of slopes on $\partial N(L)$, with r_i a slope on $\partial N(L_i)$. Then $L(r)$ denotes the r -Dehn surgery on L , which is the manifold obtained by gluing k solid tori V_1, \dots, V_k to $E(L)$ so that each r_i is identified with a meridian disk of V_i . The surgery and the slope r are *totally nontrivial* if no r_i is the meridian slope of L_i .

Corollary 2. *Let L be as in Theorem 1. If each component of L intersects some allowable sphere, then $L(r)$ is a Haken manifold for all totally nontrivial r , and the*

surfaces F and F' in Theorem 1 remains incompressible in $L(r)$.

Remark. (1) It is easy to see that $F = F(a_1, \dots, a_m)$ being incompressible implies that $P = P(a_1, \dots, a_m)$ is an essential planar surface in $E(L)$. With a similar proof to that of Theorem 1 one can show that P is essential even if the condition $|a_{ij}| \geq 3$ in (iii) of Theorem 1 is replaced by $|a_{ij}| \geq 2$. This generalizes the main theorem of [FM1].

(2) When $n \leq 2$, the link is a 2-bridge link, so by [HT] $E(L)$ contains no closed essential surface. Hence the assumption $n \geq 3$ in Theorem 1 is necessary.

(3) By definition of $2n$ -plat projection, the number of rows m is odd. If $m = 1$ the link is a composite link, and our assumption implies that it is nonsplit. In this case $E(L)$ is irreducible, and the surfaces in the theorem are swallow-follow tori, which are essential. Therefore the theorem is true for $m = 1$. We may thus assume that $m \geq 3$ in the proof of Theorem 1.

(4) In Corollary 2, each component of L intersects some allowable sphere if and only if no component of L is on the left of $S(1, \dots, 1)$ or the right of $S(n-2, n-1, \dots, n-2)$, which is equivalent to that a_{i1} and $a_{j,n-1}$ are odd for some odd i, j .

(5) The results remain true if we replace the twist tangles with rational tangles of slopes p_{ij}/a_{ij} with a_{ij} satisfying the conditions in the theorem, or certain kinds of more complicated tangles. However in this case the link diagram would not be in $2n$ -plat form.

A p/q rational tangle is a pair (B, T) , where B is a “pillow case” in \mathbf{R}^3 with corner points $(0, \pm 1, \pm 1)$, and T is obtained by taking 2 arcs of slope p/q on ∂B connecting the four corner points of the pillow case, then pushing the interior of the arcs into the interior of B . The xz -plane intersects ∂B in a circle C of slope ∞ , called a *vertical circle* on ∂B . Each component of $\partial B - C$ contains two points of ∂T . We need the following result about rational tangles.

Lemma 3. *Suppose (B, T) is a p/q rational tangle, and C a vertical circle on ∂B . Let $X = B - \text{Int}N(T)$, and let P be a component of $(\partial B \cap X) - C$.*

- (i) *if $q \geq 1$ then P is incompressible in X ;*
- (ii) *if $q \geq 2$, then $\partial X - C$ is incompressible in X ;*
- (iii) *if $q \geq 3$, then any compressing disk of ∂X intersects P at least twice.*

Proof. (ii) Notice that when attaching a 2-handle to X along the curve C , the manifold X_C is the exterior of a 2-bridge link associated to the rational number p/q , which is nontrivial and nonsplit when $q \geq 2$. In particular, ∂X_C is incompressible. If D is a compressing disk of ∂X disjoint from C , then since X is a handlebody of genus 2, we can find a nonseparating compressing disk D' which is still disjoint from C . But then D' would remain a compressing disk in X_C , a contradiction.

(i) If $q \geq 2$ this follows from (ii) and the fact that P is a subsurface of $\partial X - C$ whose complement contains no disk components. If $q = 1$, X is a product $P \times I$, and the result is obvious.

(iii) By (i) P is incompressible, which also implies that $\partial X - P$ is incompressible because any simple loop on $\partial X - P$ is isotopic to one in P . By [Wu, Lemma 2.1] there is no compressing disk of X intersecting P at a single essential arc. \square

The following lemma is well-known. The proof is an easy inner-most circle outer-most arc argument, and will be omitted.

Lemma 4. *Let F be an essential surface in a compact orientable 3-manifold M . If $M' = M - \text{Int}N(F)$ is irreducible, and no compressing disk of $\partial M'$ is disjoint from the two copies of F on $\partial M'$, then M is irreducible and ∂ -irreducible.*

We now proceed to prove Theorem 1. In the following, we will assume that L is a link as in Theorem 1. By the remark above, we may assume $m \geq 3$.

Lemma 5. *The manifold $X = X(1, \dots, 1)$ is irreducible and ∂ -irreducible.*

Proof. Consider the tangle (B, T) on the left of S . By an isotopy of (B, T) we can untwist the boxes in T which lie on the even rows of the projection of L , so the tangle (B, T) is equivalent to the one shown in Figure 2, where each box corresponds to the first box on an odd row of the projection of L ; hence there are $k = (m + 1)/2 \geq 2$ boxes, ($k = 3$ in Figure 2.) Let D_1, \dots, D_k be the disks represented by the dotted lines in Figure 2, which cuts (B, T) into $k + 1$ subtangles $(B_0, T_0), \dots, (B_k, T_k)$, where (B_0, T_0) is the one in the middle, which intersects all the D_i . Let $P_i = D_i \cap X$ be the twice punctured disk in X corresponding to D_i . They cut X into X_0, \dots, X_k , with $X_i = B_i - \text{Int}N(T_i)$ the tangle space of (B_i, T_i) .

We want to show that $\cup P_i$ is essential in X . Since each (B_i, T_i) , $i \geq 1$, is a twist tangle with at least 3 twists, by Lemma 3, the surface P_i is essential in X_i . Now consider X_0 . Put $Q = \partial B_0 - \cup D_i$. If D is a compressing disk of Q in X_0 , then it is a disk in B_0 disjoint from $T_0 \cup (\cup D_i)$; but since $T_0 \cup (\cup D_i)$ is connected, this would imply that one side of D is disjoint from all D_i , hence ∂D is a trivial curve on Q , which is a contradiction. Therefore Q is incompressible in X_0 . Assume there is a disk D in X_0 such that $\partial D \cap (\cup P_i)$ has only one component. Since each string of T_0 has ends on different D_i , we see that $\partial D \cap \partial N(T_0) = \emptyset$, so $\partial D \cap (\cup P_i)$ is either a proper arc in some D_i which separates the two points of T_0 on D_i , or it is a circle bounding a disk on D_i containing exactly one point of T_0 , or ∂D can be isotoped into Q . The first two cases are impossible because then D would be a disk in B_0 disjoint from T_0 and yet each component of $\partial B_0 - \partial D$ contains an odd number of endpoints of T_0 . The third case contradicts the incompressibility of Q . This completes the proof that $\cup P_i$ is an essential surface in X .

Notice that all X_i are handlebodies, and hence irreducible. Since Q is incompressible in X_0 , and by Lemma 3, the surfaces $\partial X_i - P_i \subset \partial X_i - \partial D_i$ are incompressible in X_i for $i \geq 1$, it follows from Lemma 4 that X is irreducible and ∂ -irreducible. \square

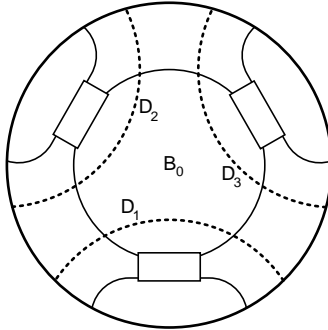


Figure 2

Lemma 6. *The manifold $X = X(a_1, \dots, a_m)$ associated to an allowable sphere $S(a_1, \dots, a_m)$ is irreducible and ∂ -irreducible.*

Proof. There is a sequence of allowable spheres S_1, \dots, S_{k+1} , such that $S_1 = S(1, \dots, 1)$, $S_{k+1} = S(a_1, \dots, a_m)$, and the non-common part of S_i, S_{i+1} bounds a single box in the projection of L , that is, $S_i \cup S_{i+1} - \text{Int}(S_i \cap S_{i+1}) = \partial \tilde{B}_i$ for some twist tangle $(\tilde{B}_i, \tilde{T}_i)$ with $a \neq 0$ left hand half-twists. Let (B_i, T_i) be the tangle on the left of S_i , and let $X_i = B_i - \text{Int}N(T_i)$. Similarly, let $\tilde{X}_i = \tilde{B}_i - \text{Int}N(\tilde{T}_i)$. Thus $X = X_{k+1} = X_k \cup_P \tilde{X}_k$, where $P = X_k \cap \tilde{X}_k$ is a twice punctured disk. By Lemma 5, X_1 is irreducible and ∂ -irreducible, and by induction on the length of the sequence we may assume that X_k is irreducible and ∂ -irreducible. Clearly P is incompressible and ∂ -incompressible on the X_k side. If $|a| \geq 3$ then, by Lemma 3, P is also incompressible and ∂ -incompressible on the \tilde{X}_k side, hence P is an essential surface in X . Since $\partial \tilde{X}_k - P$ is also incompressible in \tilde{X}_k , and since X_k and \tilde{X}_k are irreducible, it follows that $X = X_k \cup_P \tilde{X}_k$ is irreducible and ∂ -irreducible. Also, if $|a| = 1$ then \tilde{X}_k is a product $P \times I$, so $X_{k+1} \cong X_k$, and the result follows.

It remains to prove the lemma for the case $|a| = 2$. In this case there is a disk D in \tilde{X}_k which intersects P in a single arc γ , cutting (\tilde{X}_k, P) into a pair $(A \times I, A \times \partial I)$, where A is an annulus. Thus

$$X = X_k \cup_P \tilde{X}_k = (X_k \cup_{\gamma \times I} (D \times I)) \cup_{A \times \partial I} (A \times I) \cong X_k \cup_{A \times \partial I} (A \times I).$$

Since a compressing disk of $\partial(A \times I)$ intersects $A \times \partial I$ at least twice, by the same argument as above, one can show that $A \times \partial I$ is essential in X , and X is irreducible and ∂ -irreducible. \square

Proof of Theorem 1. Let F, F' be the surfaces in the theorem, isotoped slightly to be disjoint from each other. Then $F \cup F'$ cuts $E(L)$ into three parts: The component on the left of F is homeomorphic to X , the one on the right of F' is homeomorphic to X' , and the one X'' between F and F' is the union of $P \times I$ and $Q \times I$, where Q is the set of tori in $\partial E(L)$ which intersect ∂P . We have shown in Lemma 6 that X is irreducible and ∂ -irreducible, and because of symmetry, so is X' . Now X'' can be cut into $F \times I$ along some (essential) meridional annuli in $Q \times I$, hence by Lemma 4 it is irreducible and ∂ -irreducible. Since F and F' have genus at least 2, they are not boundary parallel. It follows that $F \cup F'$ is essential in X , and X is irreducible. \square

Proof of Corollary 2. Let S_1, \dots, S_k be a set of disjoint allowable spheres, so that $S_1 = S(1, \dots, 1)$, $S_k = S(n-2, n-1, \dots, n-2)$, and there is only one box of the projection of L between S_i and S_{i+1} . These spheres are similar to those in the proof of Lemma 6, except that they are now mutually disjoint, so the manifold between S_i and S_{i+1} is a product $S^2 \times I$.

Let F_i be the essential surfaces corresponding to S_i , as defined before Theorem 1, isotoped slightly so that they are disjoint from each other. Also, isotope F'_k to be disjoint from F_k . Then the set of $k+1$ surfaces $F_1, F_2, \dots, F_k, F'_k$ cuts $E(L)$ into $k+2$ components Y_0, \dots, Y_{k+1} , where Y_0 is the manifold $X(1, \dots, 1)$ on the left of F_1 , $Y_{k+1} = X'(n-2, n-1, \dots, n-2)$ is the manifold on the right of F_{k+1} , Y_k is between F_k and F'_k , and for $1 \leq i \leq k-1$, Y_i is between F_i and F_{i+1} . Since all the F_i and F'_k are essential, we see that Y_i are all irreducible and ∂ -irreducible. We need to show that the manifold \hat{Y}_i obtained from Y_i by Dehn filling on its

toroidal boundary components (if any), with slopes the corresponding subset of r , is still irreducible and ∂ -irreducible. The result will then follow by gluing the pieces together along F_i and F'_k .

Our assumption implies that Y_0 and Y_{k+1} are disjoint from $\partial E(L)$, hence $\widehat{Y}_i = Y_i$ for $i = 0, k+1$. Now Y_k is a regular neighborhood of $P \cup Q$, where $P = S_k \cap E(L)$, and Q is the set of tori in $\partial E(L)$ which intersect P . Since S is separating, each component Q_j of Q intersects ∂P at least twice, so there are two nonparallel essential annuli in Y_k , each having a boundary component on Q_j with meridional slope. Applying Menasco's theorem [Me] and Scharlemann's theorem [Sch] on each component of Q , we see that after any totally nontrivial Dehn filling on Q the manifold \widehat{Y}_k is still irreducible and ∂ -irreducible.

Now assume $1 \leq i \leq k-1$. Let (B'_i, T'_i) be the twist tangle between S_i and S_{i+1} . Notice that if the twist number a of T'_i is odd then Y_i contains no component of $\partial E(L)$, so $\widehat{Y}_i = Y_i$ and we are done. If a is even, then the tangle (B_{i+1}, T_{i+1}) on the left of S_{i+1} may contain a loop K intersecting the twist tangle (B'_i, T'_i) , so Y_i may contain a single component Q of $\partial E(L)$.

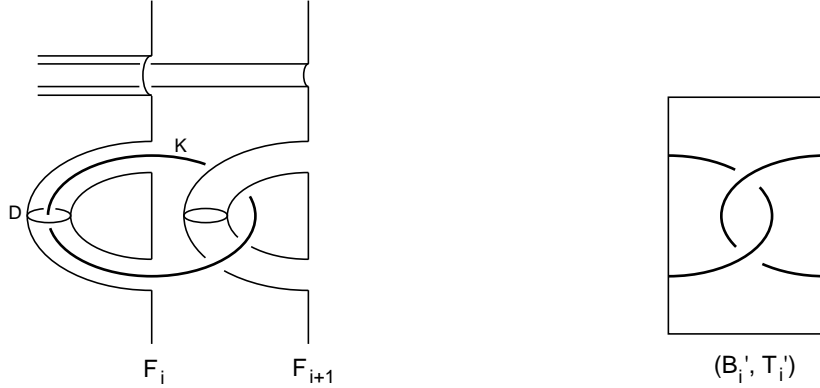


Figure 3

Let $Y_i(m)$ be the manifold obtained by the trivial Dehn filling on Q . Then F_i has a compressing disk D in $Y_i(m)$ intersecting the core K of the Dehn filling solid torus only once, so K is not a cable knot in $Y_i(m)$. See Figure 3. It follows from [Sch] that after surgery the manifold \widehat{Y}_i is irreducible. Also, by [CGLS, Theorem 2.4.3] \widehat{Y}_i is ∂ -irreducible if the surgery slope r_j on the torus Q intersects the meridional slope m at least twice. Now if r_j intersects m only once, then m is a longitude after the surgery, hence the manifold \widehat{Y}_i is homeomorphic to the one obtained by cutting Y_i along the annulus $D \cap Y_i$, denoted by \widetilde{Y}_i . Now there is an annulus A in $B_{i+1} - \text{Int} B_i$ (B_i is the ball on the left of S_i) separating the twist tangle (B'_i, T'_i) from the other arcs of L , which cuts \widetilde{Y}_i into $\widetilde{X} \cong B'_i - \text{Int} N(T'_i)$ and some $G \times I$, where G is a subsurface of F_i with one boundary component. Clearly A is essential in $G \times I$. Since the twist number a is even, our assumption in Theorem 1 implies that $|a| \geq 2$. Hence by Lemma 3 the surface $\partial \widetilde{X} - A$ is incompressible in \widetilde{X} , which implies that A is essential in \widetilde{X} . It follows that \widetilde{Y}_i is irreducible and ∂ -irreducible. \square

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